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Equilibrium Statistics of the Simplest Viscous Flow With Random Forcing

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The statistics of a 3-mode representation of randomly forced 2-dimensional viscous flow are constructed by direct numerical integration of a coupled system of nonlinear ordinary differential equations. The statistics approach equilibrium when taken over a sufficiently large ensemble of realizations and agree well with analytic solutions for statistical equilibrium.

1. INTRODUCTION

In a recent paper, Thompson (1971) has given an analytic treatment of the equilibrium statistics of a two-dimensional viscous flow with various simple types of random forcing. Expressed in terms of the stream function amplitudes A_k of a set of orthogonal functions, the evolution equations for nonlinear systems of this type are of the form

$$\alpha_k^2 \frac{dA_k}{dt} + \sum_{i=1}^{N} \sum_{j=1}^{N} \beta_{ijk} \alpha_j^2 A_i A_j + \nu \alpha_k^4 A_k = \alpha_k^2 f_k(t) (k = 1, 2, 3 \ldots N) \qquad (1)$$

where the α_k are discrete eigenvalues, ν is the kinematic coefficient of viscosity, the $f_k(t)$ are randomly varying forcing functions, and the nonlinear interaction coefficients β_{ijk} have the following properties: β_{ijk} vanishes if any two indices are the same, is invariant under cyclic permutation of indices, and reverses sign with noncyclic permutation.

One special result was that, if the random forcing is statistically isotropic in an

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581/9/1-1

N-dimensional phase space in which the coordinates are the A_k, there exists a unique stationary probability density function ρ, given by

$$\rho = Ce^{-\nu K/\mu}, \tag{2}$$

where

$$K = \tfrac{1}{2} \sum_{k=1}^{N} \alpha_k^2 A_k^2$$

and

$$\mu = D_N \sum_{k=1}^{N} \langle f_k^2 \rangle \int_0^\infty \Phi(\tau)\, d\tau.$$

The positive constant D_N depends solely on the geometry of the phase space, the crooked brackets denote the ensemble average and $\Phi(\tau)$ is the normalized autocorrelation function for the f_k. The constant C is determined by the condition that

$$C \int_1 \cdots \int_N \rho \prod_{k=1}^{N} dA_k = C \prod_{k=1}^{N} \int_{-\infty}^{\infty} e^{-(\nu/2\mu)\,\alpha_k^2 A_k^2}\, dA_k = 1.$$

As a direct consequence of Eq. (2), the stationary ensemble average of A_p^2 is

$$\langle A_p^2 \rangle = \int_1 \cdots \int_N \rho A_p^2 \prod_{k=1}^{N} dA_k$$

$$= \frac{\mu}{\nu \alpha_p^2}. \tag{3}$$

We shall refer to this simple formula later.

In the paper cited above, it was shown only that a stationary probability distribution may exist and, if it does exist, is also unique. It is also easy to show that, if an equilibrium probability distribution exists, it is stable. That is, if the probability distribution is perturbed from equilibrium by some small amount, it tends to return to equilibrium. The analysis does not, however, answer the basic and remaining question: Do the ensemble statistics for a finite number of orthogonal modes become, in fact, independent of time, when averaged over many realizations?

Since this question cannot be resolved analytically (short of finding the general time-dependent solution for the nonequilibrium probability distribution), it appeared that a direct numerical test was in order. To gain some insight, we have considered the simplest nontrivial system represented by Eq. (1), viz., that for which $N = 3$. One might intuitively expect that the lowest order system would display the least "ergodicity" and would thus be least likely to approach a state of statistical equilibrium. In this respect, the calculation of statistics of the simplest system might be regarded as the most crucial test.

2. Formulation of the Computing Problem

In the case when $N = 3$, Eq. (1) generates the system

$$\begin{aligned}
\frac{dA_1}{dt} + \frac{\beta(\alpha_3^2 - \alpha_2^2)}{\alpha_1^2} A_2 A_3 + \nu\alpha_1^2 A_1 &= f_1 , \\
\frac{dA_2}{dt} + \frac{\beta(\alpha_1^2 - \alpha_3^2)}{\alpha_2^2} A_1 A_3 + \nu\alpha_2^2 A_2 &= f_2 , \\
\frac{dA_3}{dt} + \frac{\beta(\alpha_2^2 - \alpha_1^2)}{\alpha_3^2} A_1 A_2 + \nu\alpha_3^2 A_3 &= f_3 ,
\end{aligned} \tag{4}$$

where β is a constant. It is readily verified that, if all the f's are zero and $\nu = 0$, then

$$\frac{d}{dt} \sum_{k=1}^{3} \alpha_k^2 A_k^2 = 0 \qquad \text{and} \qquad \frac{d}{dt} \sum_{k=1}^{3} \alpha_k^4 A_k^2 = 0.$$

These invariants were used to check the program for numerical time integration of Eqs. (4). If the f's are all zero and $\beta = 0$, the solutions of Eqs. (4) are just exponential functions with known e-folding times.

In order to avoid computing spurious unstable solutions, we have adopted simple one-step forward differencing in time. With a suitable choice of dimensionless variables and values of β, α_1, α_2, and α_3 appropriate to the lowest three modes for a rectangular domain whose aspect ratio is $\sqrt{2}$, the finite difference equations corresponding to Eqs. (4) are

$$\begin{aligned}
X_{n+1} &= X_n - \tau(3X_n + 3Y_nZ_n + F_n), \\
Y_{n+1} &= Y_n - \tau(6Y_n - 3X_nZ_n + G_n), \\
Z_{n+1} &= Z_n - \tau(9Z_n + X_nY_n + H_n),
\end{aligned} \tag{5}$$

in which $\tau = \pi^2 \Delta t$, Δt is the time increment, and the subscript n denotes conditions at the n-th time level. The variables X_n, Y_n, Z_n, F_n, G_n, and H_n are dimensionless forms of A_1, A_2, A_3, f_1, f_2 and f_3, respectively.

The randomly varying functions F_n, G_n and H_n were generated by performing a two-decision "random walk" on the surface of a sphere with fixed radius A in the (F_n, G_n, H_n) space. Specifically,

$$\begin{aligned}
F_{3m} &= AS_{3m} , & G_{3m} &= AC_{3m}\sigma_{3m} , & H_{3m} &= AC_{3m}\gamma_{3m} , \\
F_{3m+1} &= AC_{3m+1}\gamma_{3m+1} , & G_{3m+1} &= AS_{3m+1} , & H_{3m+1} &= AC_{3m+1}\sigma_{3m+1} , \\
F_{3m+2} &= AC_{3m+2}\sigma_{3m+2} , & G_{3m+2} &= AC_{3m+2}\gamma_{3m+2} , & H_{3m+2} &= AS_{3m+2} ,
\end{aligned}$$

for $m = 1, 2, 3, \ldots$. The C_n, S_n, σ_n, and γ_n were generated from the formulas

$$S_n^* = \lambda S_n + \mu r_n C_n,$$
$$C_n^* = \lambda C_n - \mu r_n S_n,$$
$$\sigma_n^* = \lambda \sigma_n + \mu p_n \gamma_n,$$
$$\gamma_n^* = \lambda \gamma_n - \mu p_n \sigma_n,$$

and

$$S_{n+1} = C_n^* \sigma_n^*,$$
$$C_{n+1} = (1 - C_n^{*2} \sigma_n^{*2})^{1/2},$$
$$\sigma_{n+1} = \frac{C_n^* \gamma_n^*}{C_{n+1}},$$
$$\gamma_{n+1} = \frac{S_n^*}{C_{n+1}},$$

in which λ and μ are constants, such that $0 < \lambda < 1$ and $\mu = (1 - \lambda^2)^{1/2}$. The variables r_n and p_n have the values one or minus one, chosen independently and by random choice at each time stage.

It is easily verified that, if $S_0^2 + C_0^2 = 1$ and $\sigma_0^2 + \gamma_0^2 = 1$, then

$$S_n^{*2} + C_n^{*2} = 1, \qquad \sigma_n^{*2} + \gamma_n^{*2} = 1,$$
$$S_n^2 + C_n^2 = 1, \qquad \sigma_n^2 + \gamma_n^2 = 1,$$
$$F_n^2 + G_n^2 + H_n^2 = A^2.$$

These relationships were used as checks of the program for generating the randomly varying functions F_n, G_n and H_n. Cyclic permutation of three different formulas for computing the functions F_n, G_n, and H_n insures that the forcing is statistically isotropic, since there are then an equal number of random choices involved in computing each random variable during every cycle of 3 time steps. The relations between the variables S_n, C_n, σ_n, γ_n, and S_n^*, C_n^*, σ_n^*, γ_n^* guarantee continuity of the F_n, G_n, and H_n, despite the fact that the latter are computed from formulas that change from one time stage to the next.

3. The Computations and Output

Starting with initial conditions $X_0 = Y_0 = Z_0 = 0$, Eqs. (5) were integrated numerically for $\tau = \pi^2 \Delta t = .04$. This value was small enough to suppress truncation error to within a few percent. The calculations were carried out for $\lambda = .8$

and $A = 10.0$. As a result, the random functions changed sign about once every ten time steps.

The output was presented in several forms, the simplest and most relevant of which are fixed-interval time averages and cumulative time averages of X_n^2, Y_n^2, and Z_n^2. Some types of output were diagnostic checks of the program, and another is a count of the number of time stages at which the coordinates (X_n, Y_n, Z_n) lie in the range $id < X_n \leqslant (i+1)\,d, jd < Y_n \leqslant (j+1)\,d$, and $kd < Z_n \leqslant (k+1)\,d$, for an arbitrarily chosen constant d and selected limits of the indices i, j, and k. The latter gives the numerically calculated probability density function.

A listing of the Fortran IV program is available on request.

4. Numerical Results and Interpretation

The main results are summarized in Figs. 1(a)–(c). Comparing Figs. 1(a) and 1(b), we first note that the variability of mean square values of X_n, Y_n and Z_n is much less for 1000 time-step averaging intervals than for 100 time-step intervals. Moreover, Fig. 1(c) shows that the cumulative average has become essentially stationary after about 5000 time steps. Thus, since the forcing is random with short period and the longest natural time scale is of the order of 100 time steps, we conclude that ensemble statistics, taken over about 50 or more independent realizations, are also quasistationary. This is convincing evidence that ensemble statistics taken over a sufficiently large number of realizations do indeed approach equilibrium, even if the number of degrees of freedom is small.

To show that the result of this experiment was not entirely accidental, we merely point out that the expected equilibrium ratios of the mean square values of X_n, Y_n, and Z_n are given analytically by Eq. (3). With corrections for systematic truncation error in the rate of viscous dissipation, the expected ratios are 2.15:1.00:.61. The numerically computed averages give the ratios 2.18:1.00:.55. In short, our conjectures about the existence and uniqueness of equilibrium ensemble statistics of systems governed by Eqs. (1) appear to be confirmed by numerical computation.

Some interesting, but less conclusive results are shown in Figs. 2(a)–(d), which show a contour (solid lines) of the numerically computed probability density function in the three coordinate planes. For comparison, the elliptical intersections of the coordinate planes with surfaces of constant K are drawn as dashed lines. The "ridge" between lower interior and lower exterior values of the computed probability density function is shown by a dash-dotted line.

Inspecting Figs. 2(b)–(d), we first observe that the computed probability distribution is essentially constant on the surfaces $X_n^2 + 2Y_n^2 + 3Z_n^2 = \text{constant}$, as predicted by the analytical results. Although the functional dependence on K does not correspond exactly with that given by Eq. (2), we should emphasize that the

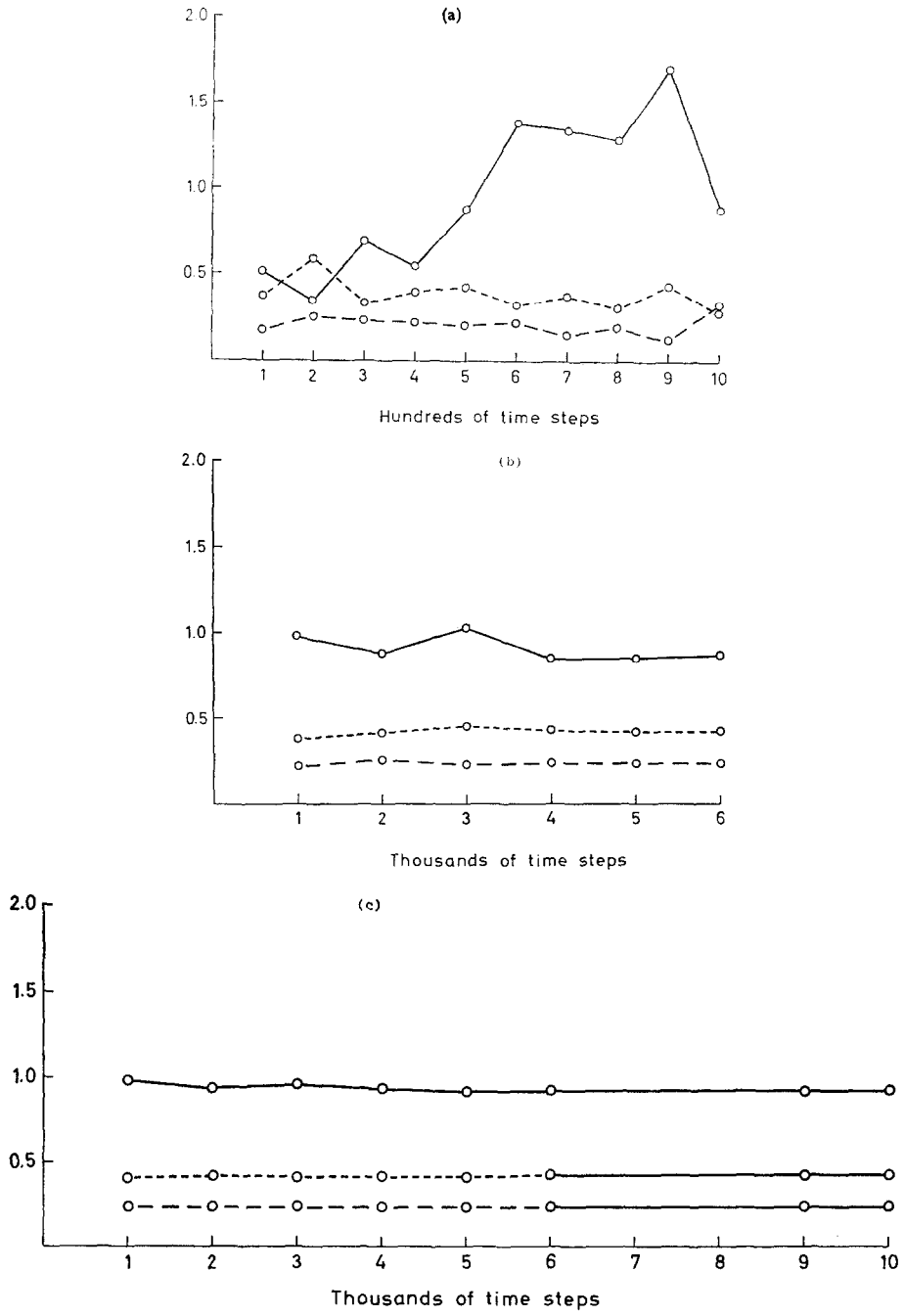

FIG. 1. Graphs of time-averaged values of X_n^2 (solid lines), Y_n^2 (dashed lines) and Z_n^2 (dash-dotted lines), in relative units.

(a) Averages over successive 100 time steps.

(b) Averages over successive 1000 time steps.

(c) Cumulative average as a function of averaging interval.

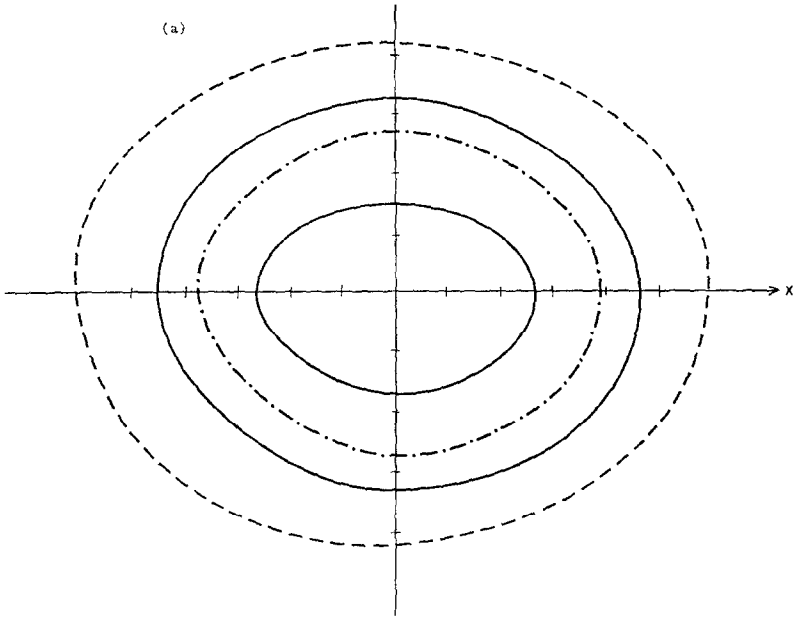

FIG. 2. Contours of the computed probability density function (solid lines) and "ridge-lines" (dash-dotted). The dashed curves are elliptical intersections of the coordinate planes with surfaces of constant K.

(a) X–Y plane at 20 000 time steps.

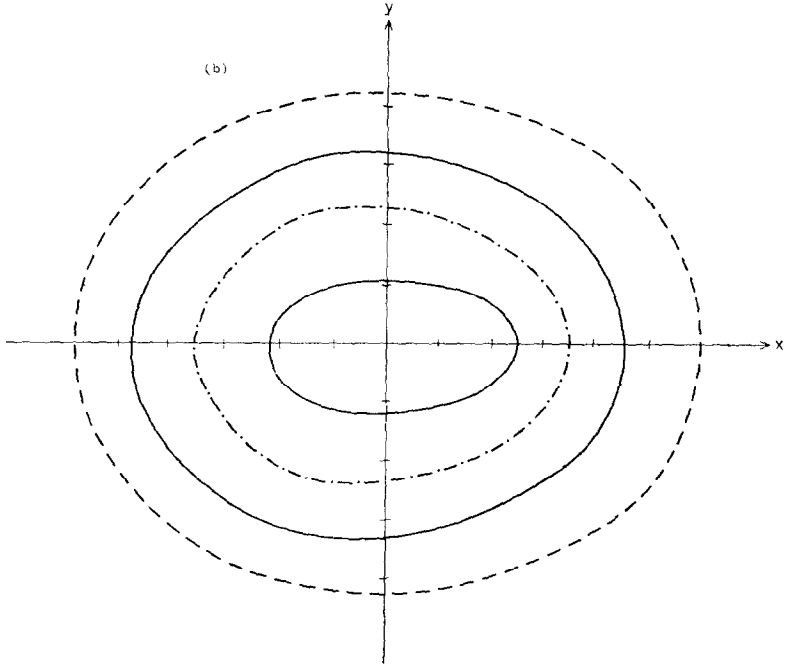

FIG. 2. (b) X–Y plane at 30 000 time steps.

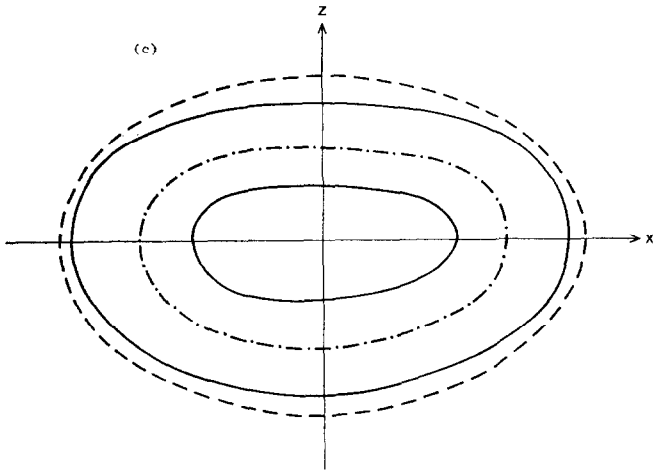

FIG. 2. (c) X–Z plane at 30 000 time steps.

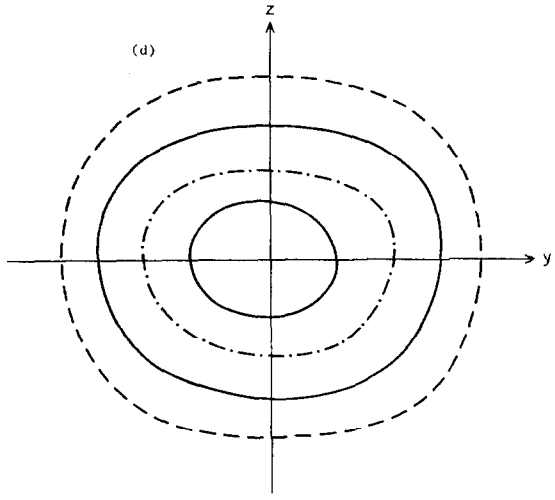

FIG. 2. (d) Y–Z plane at 30 000 time steps.

distributions shown in Fig. 2 cannot be in statistical equilibrium for a large ensemble of independent realizations but, in that sense, must tend toward the normal distribution. This follows from the continuity (or Liouville) equation for a true ensemble of points in phase space, which implies that

$$\frac{\partial}{\partial t}\int_V \rho\, dV = \nu \int_S \rho \frac{\partial K}{\partial n}\, dS + \mu \int_S \frac{\partial \rho}{\partial n}\, dS,$$

where the integrations are taken over a volume V bounded by a closed surface S on which K is constant. The derivatives with respect to n are directed normal to S and outward. Thus, since $\rho > 0$, the ensemble probability density must increase with time inside any closed shell of maximum ρ. This is, in fact, verified by comparing Figs. 2(a) and (b); the probability density increases near the origin, and the shell of maximum ρ shrinks, but very slowly.

One should expect, therefore, that the probability density would gradually peak at the origin, approaching the $e^{-\nu k/\mu}$ distribution after a very long time integration. Since machine time was short and convergence was obviously slow, the authors decided to start afresh later, but with $A = 1$, in which case the external Reynolds number would be an order of magnitude smaller and the convergence rate would have to be correspondingly increased.

Despite incomplete convergence, the probability distribution is evidently close enough to equilibrium to give stable statistics for quantities that are sampled several thousand times more often. As pointed out earlier, the computed averages of X_n^2, Y_n^2 and Z_n^2 become quasi-stationary and are very close to their theoretically derived equilibrium values.

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References

1. P. D. Thompson, "Statistical Theory of Two-Dimensional Viscous Flow With Isotropic and Simple Anisotropic Random Forcing," in press.